1 a) $\mathcal{V} = \mathbb{R}^2$, $\mathcal{U} = \operatorname{span}\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$, $\mathcal{W} = \operatorname{span}\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$.

Both $\mathcal{U}$ and $\mathcal{W}$ are subspaces of $\mathcal{V}$.

Now take $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \in \mathcal{U} \cup \mathcal{W}$.

Since $\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \notin \mathcal{U} \cup \mathcal{W}$,

$\mathcal{U} \cup \mathcal{W}$ is not a subspace of $\mathcal{V}$.

b) If $\mathcal{U} \subseteq \mathcal{W}$ then $\mathcal{U} \cup \mathcal{W} = \mathcal{W}$ so $\mathcal{U} \cup \mathcal{W}$ is a subspace.

Similarly, if $\mathcal{W} \subseteq \mathcal{U}$ then $\mathcal{U} \cup \mathcal{W} = \mathcal{U}$ so $\mathcal{U} \cup \mathcal{W}$ is a subspace.

c) Assume that $\mathcal{U} \cup \mathcal{W}$ is a subspace of $\mathcal{V}$.

Suppose that there exists a vector $u \in \mathcal{U}$ s.t. $u \notin \mathcal{W}$.

Let $w \in \mathcal{W}$. Then $w \in \mathcal{U} \cup \mathcal{W}$. Also, $u \in \mathcal{U} \cup \mathcal{W}$.

Since $\mathcal{U} \cup \mathcal{W}$ is a subspace, $u + w \in \mathcal{U} \cup \mathcal{W}$.

We conclude that $u + w \in \mathcal{U}$ or $u + w \in \mathcal{W}$.

d) Assume that $\mathcal{U} \cup \mathcal{W}$ is a subspace of $\mathcal{V}$.
If $\mathcal{U} \subseteq \mathcal{W}$ then we are done.
Suppose $\mathcal{U} \not\subseteq \mathcal{W}$. Then there exists a $u \in \mathcal{U}$ such that $u \notin \mathcal{W}$.
Let $w \in \mathcal{W}$. By c), either $u+w \in \mathcal{U}$ or $u+w \in \mathcal{W}$.

If $u+w \in \mathcal{W}$ then $\underbrace{u+w}_{\in \mathcal{W}} \underbrace{-w}_{\in \mathcal{W}} = u \in \mathcal{W}$
since $\mathcal{W}$ is a subspace. This, however, is a contradiction by .

We must therefore have that $u+w \in \mathcal{U}$.
But then $\underbrace{u+w}_{\in \mathcal{U}} \underbrace{-u}_{\in \mathcal{U}} = w \in \mathcal{U}$, since
$\mathcal{U}$ is a subspace. Because $w \in \mathcal{W}$ was arbitrary, we conclude that $\mathcal{W} \subseteq \mathcal{U}$. $\square$

2 a) $T : P_2 \to P_2$ is defined by

$$(T(p))(x) = x^2 p\left(\frac{1}{x}\right) + p''(x).$$

Let $p, q \in P_2$ and $a, b \in \mathbb{R}$. To show:
$T(ap+bq) = aT(p) + bT(q)$.

Write $p(x) = c_0 + c_1 x + c_2 x^2$
$q(x) = d_0 + d_1 x + d_2 x^2$
where $c_i, d_i \in \mathbb{R}$ for $i = 0, 1, 2$.

$$T(p) = x^2\left(c_0 + c_1 \frac{1}{x} + c_2 \frac{1}{x^2}\right) + 2c_2$$
$$= 3c_2 + c_1 x + c_0 x^2.$$

Similarly,
$$T(q) = 3d_2 + d_1 x + d_0 x^2$$

Now, compute $T(ap+bq)$
$$= T\left(a(c_0+c_1x+c_2x^2) + b(d_0+d_1x+d_2x^2)\right)$$
$$= T\left(ac_0 + bd_0 + (ac_1+bd_1)x + (ac_2+bd_2)x^2\right)$$
$$= 3(ac_2+bd_2) + (ac_1+bd_1)x + (ac_0+bd_0)x^2$$
$$= a(3c_2 + c_1x + c_0x^2) + b(3d_2 + d_1x + d_0x^2)$$
$$= aT(p) + bT(q).$$

So $T$ is a linear operator.

b) $T(1) = x^2 = -3 \cdot 1 + 2(1+x) + 1 \cdot (1-2x+x^2)$

$T(1+x) = x^2\left(1+\frac{1}{x}\right) = x + x^2$

$= -4 \cdot 1 + 3 \cdot (1+x) + 1 \cdot (1-2x+x^2)$

$T(1-2x+x^2) = x^2\left(1 - \frac{2}{x} + \frac{1}{x^2}\right) + 2$

$= 3 - 2x + x^2$

$= 2 \cdot 1 + 0 \cdot (1+x) + 1 \cdot (1-2x+x^2)$

So the matrix representation of $T$ with respect to $(1, 1+x, 1-2x+x^2)$ is:

$$\begin{pmatrix} -3 & -4 & 2 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

c) $R$ maps the polynomial $x^2$ to $\frac{1}{x} + 2$, which is not in $P_2$. So $R$ is not a mapping from $P_2$ to itself and therefore not a linear operator on $P_2$.

$S(x^2) = 4$, while $S(x^2 + x^2) = 32$.

So $S(x^2 + x^2) \neq S(x^2) + S(x^2)$.

Thus, $S$ is not a linear operator.